Eigenvectors and eigenvalues

There is an important aspect of matrix theory that arises from the fact that matrices can be used to represent transformations. If a vector **v** is transformed by a matrix **A**, then the resulting vector is **Av**. If the direction of the vector is unchanged once it has been transformed we can write:

$$\mathbf{Av} = \lambda\mathbf{v}$$

where λ is a constant. The vector **v** is called an 'eigenvector' of the matrix, and the corresponding value of λ is called an 'eigenvalue'.

To find the eigenvectors and eigenvalues, you have to work with the equation $\mathbf{Av} = \lambda\mathbf{v}$:

$$\mathbf{Av} = \lambda\mathbf{v} \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

where **0** is the zero matrix. Notice that we had to insert the identity matrix **I** into the equation since we cannot subtract a scalar from a matrix.

This equation would be true if either $(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$, or $\mathbf{v} = \mathbf{0}$, but these are not going to be very helpful since this is just the trivial solution and is not the one we are interested in.

To solve a general matrix equation $\mathbf{Bx} = \mathbf{C}$, we would normally find the inverse of **B** and calculate $\mathbf{x} = \mathbf{B}^{-1}\mathbf{C}$. Using this technique to try to solve the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$**,** then if $(\mathbf{A} - \lambda\mathbf{I})$ has an inverse we get $\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})^{-1}\mathbf{0}$, *ie* $\mathbf{v} = \mathbf{0}$. Since we are looking for a non-trivial solution, we must prevent this method from working. The only thing that would stop us getting $\mathbf{v} = \mathbf{0}$ would be if $\mathbf{A} - \lambda\mathbf{I}$ does not have an inverse *ie* it is singular. Remembering that singular matrices have a determinant of zero this gives us a way to find the eigenvalues and eigenvectors.

In summary, to find the eigenvalues of a matrix **A** we must solve the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. The equation obtained from $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is called the 'characteristic equation of the matrix'.

Once you have the eigenvalues, you can return to the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, to find the eigenvectors.

This method will work for a general $n \times n$ matrix, but due to the complication of calculating the determinant we will only look at 2×2 and 3×3 matrices here.

Example

Find the eigenvectors and corresponding eigenvalues of the matrix $\mathbf{B}=\begin{pmatrix}2 & -1\\ 2 & 5\end{pmatrix}$.

Solution

First we use the equation $\det(\mathbf{B}-\lambda\mathbf{I})=0$ to find the eigenvalues:

$$\det\begin{pmatrix}2-\lambda & -1\\ 2 & 5-\lambda\end{pmatrix}=0$$

$$(2-\lambda)(5-\lambda)+2=0$$

$$\lambda^2-7\lambda+12=0$$

$$(\lambda-3)(\lambda-4)=0$$

so $\lambda=3$ or $\lambda=4$.

We now use the equation $(\mathbf{B}-\lambda\mathbf{I})\mathbf{v}=0$ to find the eigenvectors, letting $\mathbf{v}=\begin{pmatrix}x\\ y\end{pmatrix}$. There is a separate eigenvector corresponding to each eigenvalue.

If $\lambda=3$, then $\begin{pmatrix}2-\lambda & -1\\ 2 & 5-\lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$, *ie* $\begin{pmatrix}-1 & -1\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$, *ie* $\begin{pmatrix}-x-y\\ 2x+2y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$. Any value of x and y such that $x+y=0$ *ie* $y=-x$ will make both rows work out.

So the eigenvector corresponding to $\lambda=3$ is $k\begin{pmatrix}1\\ -1\end{pmatrix}$, where k is a constant.

If $\lambda=4$, then $\begin{pmatrix}-2 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$, so the eigenvector corresponding to $\lambda=4$ is $k\begin{pmatrix}1\\ -2\end{pmatrix}$.

Notice that we include k in the eigenvector, since any value of the constant k will give an eigenvector.

Question 1.1

Find the eigenvectors and corresponding eigenvalues of the matrix $\mathbf{B}=\begin{pmatrix}1 & 2\\ 3 & -4\end{pmatrix}$.

Question 1.2

Show that $\lambda=1$ is an eigenvalue of the matrix $\begin{pmatrix}2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{pmatrix}$, and hence find all the eigenvalues and eigenvectors.

You are given that $\lambda^3-2\lambda^2-5\lambda+6=(\lambda-1)(\lambda^2-\lambda-6)$.

Solutions

Solution 1.1

For eigenvalues, $\det\begin{pmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{pmatrix}=0$, so we have the characteristic equation:

$$(1-\lambda)(-4-\lambda)-6=0 \Rightarrow \lambda^2+3\lambda-10=0$$

This gives $\lambda=2,\ \lambda=-5$.

If $\lambda=2$, then $\begin{pmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So the eigenvector corresponding to $\lambda=2$ is $k\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

If $\lambda=-5$, then $\begin{pmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So the eigenvector corresponding to $\lambda=-5$ is $k\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

Solution 1.2

For the eigenvalues we must solve the equation:

$$\det\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix} = 0$$

So we have the equation:

$$(2-\lambda)\{(1-\lambda)(-1-\lambda)-3\}+2\{(-1-\lambda)-1\}+3\{3-(1-\lambda)\}=0$$

and this equation simplifies to $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$.

Factorising the given cubic equation completely gives:

$$(\lambda-1)(\lambda-3)(\lambda+2)=0$$

The eigenvalues are 1, 3, and –2.

When $\lambda = 1$, $\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which gives the eigenvector of $k\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

When $\lambda = 3$, $\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which gives the eigenvector of $k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

When $\lambda=-2$, $\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which gives the eigenvector of $k\begin{pmatrix} 11 \\ 1 \\ -14 \end{pmatrix}$.